Lecture 12 **2012/12/17**

Outline

1. **Motivation**
2. **Lyapunov Stability of Autonomous Systems**
3. **Lyapunov Stability of Linear Systems with Constant Coefficients**
4. **Summary**

1. Motivation

- Lyapunov method is one of the most important tools in nonlinear systems. It is extremely important in analysis of stability of autonomous systems. It works not only for the local, but also for the global.
- Lyapunov method has a widespread use in mathematics, control sciences, engineering, physics, etc.

2. Lyapunov Stability of Autonomous Systems

Consider an autonomous system

$$\dot{x} = f(x), \tag{12.1}$$

where $f : D \to R^n$ is locally Lip., $D \subseteq R^n$ and $f(0) = 0$.

We say that system (12.1) is **complete** if for any initial state $x_0 \in D$, the solution $x = x(t; x_0)$ of (12.1) exists for all $t \geq 0$. That is, there is no blow-up for any $x_0 \in D \subseteq R^n$.

1) Statement of Lyapunov Theorem for AS

Theorem 12.1 Let $V : D \to R$ be of C^1 such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}; \tag{12.2a}$$

$$\dot{V}(x) \leq 0 \text{ in } D. \tag{12.2b}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\}, \tag{12.2c}$$

then $x = 0$ is asymptotically stable.

Remark 12.1 If $V(x)$ satisfies (12.2a), it is said **positive definite**.

2) Interpretation of Lyapunov conditions

Fact 1. If $V(x) > 0$, then there exists $c^* > 0$ such that for all $c \in (0, c^*)$, then the set $V_c := \{x \in R^n \mid V(x) = c\}$ is a compact set encircling the origin – which is said to be **a Lyapunov surface**;

Remark 12.2 If $V(x) > 0$, we can't conclude that for any $c > 0$, V_c is compact. For example, $V(x_1, x_2) = x_2^2 + \frac{x_1^2}{1 + x_1^2} > 0$; for $0 < c < 1$, $V(x_1, x_2) = c$ is compact; and for $c > 1$, $V(x_1, x_2) = c$ is not compact. See Fig. 12.1

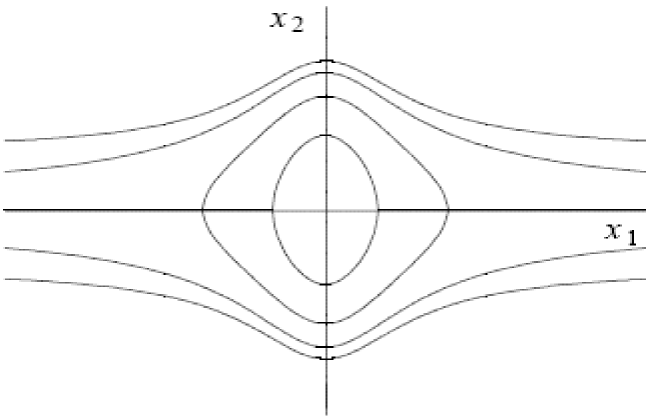

Fig. 12.1

Fact 2. The derivative of $V(x)$ along trajectories of the system (12.1) is:

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot f(x),$$

which is an inner product of the gradient of V, **a normal direction** at x of a Lyapunov surface, and $f(x)$, **a tangent direction** at the same point x of the same Lyapunov surface; i.e. **cosine of the included angle** of such two particular vectors.

Fact 3. If $\dot{V}(x) < 0$, i.e. cosine of the included angle of above two particular vectors is within $(-\frac{\pi}{2}, 0)$, then, the trajectories move inside the Lyapunov surface V_c.

3) Proof of Lyapunov Theorem for AC

Stability:

Step 1. Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that

$$B_r = \{x \in R^n \mid \|x\| \leq r\} \subset D.$$

Step 2. Let $\alpha = \min_{\|x\|=r} V(x)$. Then, $\alpha > 0$ since $V(x) > 0$, $x \neq 0$. Take $\beta \in (0, \alpha)$, and let

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}.$$

Then, Ω_β is in the interior of B_r by definition.

Step 3. Ω_β is invariant because $V(x(t)) \leq V(x_0) \leq \beta$ for all $t \geq 0$ by (12.2b).

Step 4. Ω_β is compact because it is closed by definition and bounded since $\Omega_\beta \subset B_r$. Hence, the system (12.1) has a unique solution for all $t \geq 0$ whenever $x_0 \in \Omega_\beta$ by Continuation Theorem.

Step 5. Since $V(x)$ is continuous and $V(0) = 0$, there is $\delta > 0$ such that

$$\|x\| \leq \delta \;\Rightarrow\; V(x) < \beta.$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x_0 \in B_\delta \;\Rightarrow\; x_0 \in \Omega_\beta \;\Rightarrow\; x(t) \in \Omega_\beta \;\Rightarrow\; x(t) \in B_r.$$

Therefore, for any given $\varepsilon > 0$, there exists $\delta > 0$ s.t.

$$\|x_0\| < \delta \;\Rightarrow\; \|x(t)\| < r \leq \varepsilon, \text{ for all } t \geq 0.$$

$x = 0$ is stable by definition. See Fig. 12.2.

Attractivity:

To show $\lim_{t \to +\infty} x(t) = 0$, we need to show that for any $\varepsilon > 0$, there exists $T > 0$ s.t. $\|x(t)\| \leq \varepsilon$ for all $t \geq T$.

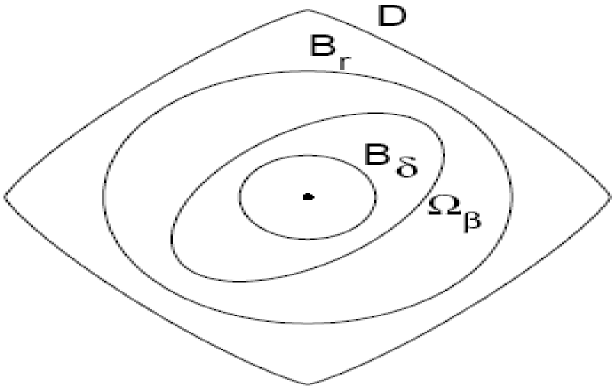

Fig. 12.2

In the proof of stability, we have shown that for any $\varepsilon > 0$ with B_ε in D, there exists $\eta > 0$ such that $\Omega_\eta \subset B_\varepsilon$ and Ω_η is invariant.

For any $x_0 \in \Omega_\eta$, $x(t) \in \Omega_\eta$ for all $t \geq 0$. Since $V(x(t))$ is monotonically decreasing by (12.2c) and bounded below by zero. Therefore, $\lim_{t\to+\infty} V(x(t)) = c \geq 0$ exists.

Now we show $c = 0$. Otherwise, we suppose $c > 0$. Since $V(x)$ is continuous, there exists $d > 0$ s.t. $B_d \subset \Omega_c \subseteq \Omega_\eta$. Then, since $\lim_{t\to+\infty} V(x(t)) = c > 0$, there exists $T > 0$ such that $x(t)$ lays outside the ball B_d for all $t \geq T$.

Let $-\gamma = \max_{d \leq \|x\| \leq \eta} \dot{V}(x)$, which exists because $\dot{V}(x)$ is continuous and has a maximum over the compact set $\{d \leq \|x\| \leq \eta\}$. By (12.2c), $-\gamma < 0$, and this implies

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\,ds \leq V(x_0) - \gamma t, \quad t \geq T.$$

For $t >> T$, $V(x(t)) < 0$. This contradicts $V(x) \geq 0$ for all x. Therefore, we have $\lim_{t\to+\infty} V(x(t)) = 0$, which implies $\lim_{t\to+\infty} x(t) = 0$ by (12.2a). □

Remark 12.3 Theorem 12.1 is a local result. For the global, we need an additional condition to make sure that Lyapunov surface $V_c := \{x \in R^n \mid V(x) = c\}$ is compact (closed and bounded in R^n) for any $c > 0$ without the broken.

4) Lyapunov Theorem for GAS

Theorem 12.2 (Barbashin-Krasovskii Theorem) Let $V: R^n \to R$ be of C^1 such that

$$V(0)=0 \text{ and } V(x)>0, \ \forall\, x \neq 0; \tag{12.3a}$$

$$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty; \tag{12.3b}$$

$$\dot{V}(x)<0, \ \forall\, x \neq 0. \tag{12.3c}$$

Then, $x=0$ is globally asymptotically stable (GAS in short).

Proof. For any $x \in R^n$, let $c=V(x)$. The condition (12.3b) implies that for such a $c>0$, there exists $r>0$ s.t. $V(x)>c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$, which implies that Ω_c is bounded. Then, the rest of the proof is similar to that of Theorem 12.1. □

Remark 12.4 The condition (12.3b) is said to be **radially unbounded**. If (12.1) is GAS, then, the equilibrium must be unique. **(why!)**

Remark 12.5 Ω_c plays an important role in analysis of Lyapunov theory. It can be found application in estimates of region of attraction.

5) Lyapunov Theorem for Unstability

Two facts:

Fact 1. $V(x)>0$ plus $\dot{V}(x)>0 \ \Rightarrow$ the origin is unstable.

Fact 2. When testing instability, the above conditions can be relaxed.

Theorem 12.3 (Chetaev Theorem) Let $V: D \to R$ be of C^1 s.t. $V(0)=0$ and $V(x_0)>0$ for some x_0 with arbitrarily small $\|x_0\|$. Define $U=\{x \in B_r \mid V(x)>0\}$ and suppose that $\dot{V}(x)>0$ in U. Then, $x=0$ is unstable.

Proof. Since $V(x_0)=a>0$, so $x_0 \in U$. $x(t)$ starting at $x(0)=x_0$ will leave U. To see this point, if $x(t) \in U \ \Rightarrow \ V(x(t)) \geq a$, since $\dot{V}(x)>0$ in U. Let

$$\gamma = \min\{\dot{V}(x) \mid x \in U \text{ and } V(x) \ge a\},$$

which exists since the continuous function $\dot{V}(x)$ has a minimum over the compact set $\{x \in U \text{ and } V(x) \ge a\}$. Then, $\gamma > 0$ and

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \ge a + \int_0^t \gamma ds = a + \gamma t .$$

This inequality shows that $\lim_{t\to+\infty} V(x(t)) = \infty$. It implies that $x(t)$ cannot stay forever in U because $V(x)$ is bounded on U. Now, $x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \ge a$ for all $t \ge 0$. Hence, it must leave U through the sphere $\|x\| = r$. Since it can happen for an arbitrarily small $\|x_0\|$, the origin is unstable. □

Remark 12.6 Since U is not necessarily a neighborhood of the origin, then

1) $V(x)$ in Chetaev Theorem does not have to be positive definite!

2) $\dot{V}(x)$ in Chetaev Theorem does not have to be positive definite!

6) Examples

Example 12.1 Consider

$$\begin{cases} x_1' = ax_1 - x_2 + kx_1(x_1^2 + x_2^2) \\ x_2' = x_1 - ax_2 + kx_1(x_1^2 + x_2^2) \end{cases}, \tag{12.4}$$

where $a > 0$, $a \ne 1$, and k is a parameter. Clearly, the origin is equilibrium. The linearization gives

$$A = \begin{pmatrix} a & -1 \\ 1 & -a \end{pmatrix}$$

with $\lambda = \pm\sqrt{a^2 - 1}$ and $g_1(x_1, x_2) = g_2(x_1, x_2) = kx_1(x_1^2 + x_2^2)$ satisfying

$$\lim_{\|x\|\to\infty} \frac{\|g(x)\|}{\|x\|} = 0 . \tag{12.5}$$

If $a > 1$, the origin is a saddle point, which is unstable. Then, (12.4) is also unstable by linearization.

If $0 < a < 1$, the origin is a center, which is stable but not AS. The linearization fails this time. However, the linearized system has the equation for trajectories given by

$$\frac{dx_2}{dx_1}=\frac{x_1-ax_2}{ax_1-x_2},$$

whose general solution is solved by

$$x_1^2-2ax_1x_2+x_2^2=\tilde{c}.$$

These trajectories are ellipses if $\tilde{c}>0$. The trajectory is the origin if $\tilde{c}=0$ (**See Remark 12.7**). So it can be taken as a Lyapunov function candidate for (12.4)

$$V(x_1,x_2)=x_1^2-2ax_1x_2+x_2^2,$$

which is positive definite. Taking derivative along trajectories results in

$$\dot{V}(x_1,x_2)=2k(x_1^2+x_2^2)(x_1^2+x_2^2-2ax_1x_2).$$

Then, $\dot{V}(x_1,x_2)<0$ if $k<0$ and $\dot{V}(x_1,x_2)>0$ if $k>0$. We conclude that (12.4) is AS if $k<0$ by Theorem 12.1 and it is unstable if $k>0$ by Theorem 12.3. Moreover, (12.4) is GAS because $V_c(x_1,x_2)=\{(x_1,x_2)\,|\,x_1^2-2ax_1x_2+x_2^2=\tilde{c}>0\}$ is a Lyapunov surface of ellipses for all $\tilde{c}>0$, which clearly satisfy the radially unbounded condition (12.3b).

Remark 12.7 The general conic equation is given by

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0, \tag{12.6}$$

where A, B and C are not all zero. If $\Delta_1\cdot\Delta_3<0$ and $\Delta_2>0$, then (12.6) is an ellipse, where

$$\Delta_1=A+C,\quad \Delta_2=\begin{vmatrix}A & B\\ C & D\end{vmatrix},\quad \Delta_3=\begin{vmatrix}A & B & D\\ B & C & E\\ D & E & F\end{vmatrix}.$$

In Example 12.1, $A=1$, $B=-a$, $C=1$, $D=0$, $E=0$ and $F=-\tilde{c}$. It is easy to be verified that when $0<a<1$, $x_1^2-2ax_1x_2+x_2^2=\tilde{c}$ for all $\tilde{c}>0$ are ellipses.

Example 12.2 Consider the pendulum equation without friction:

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-\dfrac{g}{l}\sin x_1\end{cases}.$$

Take the energy function

$$V(x)=\frac{g}{l}(1-\cos x_1)+\frac{1}{2}x_2^2.$$

Clearly, $V(0)=0$ and $V(x)>0$ is over $-2\pi < x_1 < 2\pi$.

$$\dot{V}(x)=\frac{g}{l}\dot{x}_1\sin x_1 + x_2\dot{x}_2 = \frac{g}{l}x_2\sin x_1 - \frac{g}{l}x_2\sin x_1 = 0 .$$

$\Rightarrow$ The origin is stable. Since $\dot{V}(x)\equiv 0$, $\Rightarrow$ $V(x(t))\equiv c>0$ $\Rightarrow$ $\lim_{t\to\infty} x(t)\neq 0$, the origin is not AS.

Example 12.3 Consider the pendulum equation with friction:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{m}x_2 \end{cases} . \tag{12.7}$$

Let us try again $V(x)=\frac{g}{l}(1-\cos x_1)+\frac{1}{2}x_2^2$. Since

$$\dot{V}(x)=\frac{g}{l}\dot{x}_1\sin x_1 + x_2\dot{x}_2 = -\frac{k}{m}x_2^2 \le 0 ,$$

$\Rightarrow$ The origin is stable only. However, the experience tells that it is AS because of the friction.

Remark 12.8 We may apply the finer Lyapunov function to (12.7) as follows.

$$V(x)=\frac{1}{2}x^T P x + \frac{g}{l}(1-\cos x_1) = \frac{1}{2}(x_1, x_2)\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{g}{l}(1-\cos x_1) ,$$

where P is positive definite. Try to determine the elements p_{ij} of P such that $\dot{V}(x)<0$. (**Homework**).

Example 12.4 Consider the system

$$\begin{cases} \dot{x}_1 = x_1 + g_1(x) \\ \dot{x}_2 = -x_2 + g_2(x) \end{cases} ,$$

where $g_1(x)$ and $g_2(x)$ satisfy

$$| g_j(x) | \le k \| x \|^2$$

near the origin. Consider the function

$$V(x)=\frac{1}{2}(x_1^2 - x_2^2) .$$

On the line $x_2 = 0$, $V(x)>0$ at points arbitrarily close to the origin. The derivative of $V(x)$ is given by

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x).$$

The magnitude of the term $x_1 g_1(x) - x_2 g_2(x)$ satisfies the inequality

$$|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{j=1}^{2} |x_j| \cdot |g_j(x)| \le 2k \|x\|^3.$$

Hence,

$$\dot{V}(x) \ge \|x\|^2 - 2k\|x\|^3 = \|x\|^2 (1 - 2kr\|x\|).$$

Choosing r such that $B_r \subset D$ and $r < \dfrac{1}{2k}$, all the conditions of Chetaev Theorem are satisfied and the origin is unstable.

3. Lyapunov Stability of Linear Systems with Constant Coefficients

Consider the linear system as follows

$$\dot{x} = Ax. \tag{12.8}$$

If all eigenvalues of A satisfy $\mathrm{Re}\,\lambda_j < 0$, A is said a **Hurwitz** matrix.

1) Lyapunov Method

Consider a **quadratic Lyapunov function** candidate

$$V(x) = x^T P x,$$

where P is a real symmetric positive definite matrix. The derivative of V along the trajectories of (12.8) is given by

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\Lambda}{=} -x^T Q x,$$

where Q is a symmetric matrix defined by

$$PA + A^T P = -Q. \tag{12.9}$$

If Q is positive definite, we can conclude by Theorem 12.1 that the origin is AS

Remark 12.9 (12.9) is called a Lyapunov equation of the system (12.8).

Theorem 12.4 A is Hurwitz if and only if for any given positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P that satisfies (12.9). Moreover, P is unique for each given Q in (12.9).

Proof. ($\Leftarrow$) Done.

($\Rightarrow$) If A is Hurwitz, define

$$P = \int_0^\infty \exp(A^T t) Q \exp(At)\, dt\,. \tag{12.10}$$

This integral (12.10) is well defined because A is Hurwitz. $P^T = P$ by definition. To show P is positive definite, we use contradiction. If there were $x \neq 0$ such that $x^T P x = 0$. Then,

$$x^T P x = 0 \;\;\Rightarrow\;\; \int_0^\infty x^T \exp(A^T t) Q \exp(At)\, x\, dt = 0 \;\;\Rightarrow\;\; \exp(At)x \equiv 0,\;\; \forall t \geq 0$$

$$\Rightarrow \;\; x = 0\,.$$

This contradiction shows that P is positive definite. Since

$$PA + A^T P = \int_0^\infty \exp(A^T t) Q \exp(At) A dt + \int_0^\infty A^T \exp(A^T t) Q \exp(At) dt$$

$$= \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) dt = \exp(A^T t) Q \exp(At) |_0^\infty = -Q\,,$$

which shows that P is solution of (12.9). To show uniqueness, suppose there is another solution $\tilde{P} \neq P$. Then,

$$(P - \tilde{P}) A + A^T (P - \tilde{P}) = 0\,.$$

Pre-multiplying by $\exp(A^T t)$ and post-multiplying by $\exp(At)$, we obtain

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T (P - \tilde{P})]\exp(At) = \frac{d}{dt} \exp(A^T t)(P - \tilde{P}) \exp(At)\,.$$

Hence,

$$\exp(A^T t)(P - \tilde{P}) \exp(At) \equiv \text{constant}, \;\; \forall\, t > 0\,.$$

In particular, since $\exp(At)|_{t=0} = I$, we have

$$P - \tilde{P} = \exp(A^T t)(P - \tilde{P}) \exp(At) |_{t=0} \equiv \exp(A^T t)(P - \tilde{P}) \exp(At) \to 0 \;\;\text{as}\;\; t \to \infty\,.$$

Therefore, $\tilde{P} = P$. □

2) Linearization by Lyapunov Method

Let us go back to the nonlinear system

$$\dot{x} = f(x)\,, \tag{12.11}$$

where $f: D \to R^n$ is C^1 and $f(0)=0$. Then we write (12.11) as

$$f(x) = Ax + g(x), \tag{12.12}$$

where

$$A = Df(0), \text{ and } \lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

Theorem 12.5 (Linearization)

1. The origin of (12.12) is AS if $\operatorname{Re}\lambda_j < 0$ for all eigenvalues of A.

2. The origin of (12.12) is unstable if $\operatorname{Re}\lambda_j > 0$ for one or more of the eigenvalues of A.

Proof. Let A be a Hurwitz matrix. Then, by Theorem 12.4, for any positive definite symmetric matrix Q, the solution P of the Lyapunov equation (12.9) is positive definite. We use

$$V(x) = x^T P x$$

as a Lyapunov function candidate for (12.12). The derivative of $V(x)$ along the trajectories of (12.12) is given by

$$\dot{V}(x) = x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px = x^T(PA + A^T p)x + 2x^T Pg(x)$$

$$= -x^T Qx + 2x^T Pg(x).$$

Since $\lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} = 0$, there exists $r > 0$ for any given $\gamma > 0$ such that

$$\|g(x)\| < \gamma\|x\|, \quad \forall \|x\| < r.$$

Hence,

$$\dot{V}(x) < -x^T Qx + 2\gamma\|P\|\|x\|^2, \quad \forall \|x\| < r,$$

but

$$-x^T Qx \geq \lambda_{\min}(Q)\|x\|^2$$

Note that $\lambda_{\min}(Q)$ is real and positive since Q is symmetric and positive definite. Thus

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma\|P\|]\|x\|^2, \quad \forall \|x\| < r.$$

Choosing $\gamma < \dfrac{\lambda_{\min}(Q)}{2\|P\|}$ ensures that $\dot{V}(x)$ is negative definite. By Theorem 12.1, we conclude that the origin of (12.12) is AS.

To prove the second part of the theorem, let us consider first the special case when A has no eigenvalues on the imaginary axis. Then there exists an invertible matrix T such that

$$TAT^{-1} = \begin{pmatrix} -A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 and A_2 are Hurwitz matrices. The change of variables

$$z = Tx = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

transforms (12.12) into the form

$$\begin{cases} \dot{z}_1 = -A_1 z_1 + g_1(x) \\ \dot{z}_2 = -A_2 z_2 + g_2(x) \end{cases},$$

where $g_j(z)$ satisfies

$$|g_j(z)| < \gamma \|z\|, \quad \forall \|z\| \le r, \quad j = 1, 2.$$

Let Q_1 and Q_2 be positive definite symmetric matrices. Solving

$$P_j A_j + A_j^T P_j = -Q_j, \quad j = 1, 2,$$

yields a unique positive definite solutions P_1 and P_2. Let

$$V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2 = z^T \begin{pmatrix} P_1 & 0 \\ 0 & -P_2 \end{pmatrix} z.$$

In the subspace $z_2 = 0$, $V(z) > 0$ at points arbitrarily close to the origin. Let

$$U = \{ z \in R^n \mid \|z\| \le r \text{ and } V(z) > 0 \}.$$

In U,

$$\dot{V}(z) = -z_1^T (P_1 A_1 + A_1^T P_1) z_1 + 2 z_1^T P_1 g_1(z) - z_2^T (P_2 A_2 + A_2^T P_2) z_2 - 2 z_2^T P_2 g_2(z)$$

$$= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + 2 z^T \begin{pmatrix} P_1 g_1(z) \\ -P_2 g_2(z) \end{pmatrix}$$

$$\geq \lambda_{\min}(Q_1)\|z_1\|^2 + \lambda_{\min}(Q_2)\|z_2\|^2 - 2\|z\|\sqrt{\|P_1\|^2\|g_1(z)\|^2 + \|P_2\|^2\|g_2(z)\|^2}$$

$$> (\alpha - 2\sqrt{2}\,\beta\gamma)\|z\|^2,$$

where

$$\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}. \quad \beta = \max\{\|P_1\|, \|P_2\|\}.$$

Thus, choosing $\gamma < \frac{\alpha}{2\sqrt{2}\beta}$ ensures that $\dot{V}(z) > 0$ in U. Therefore, by Theorem 12.3, the origin of (12.12) is unstable.

Remark 12.10 Notice that we could have applied Theorem 12.3 in the original coordinates by defining the matrices

$$P = T^T \begin{pmatrix} P_1 & O \\ O & -P_2 \end{pmatrix} T \quad \text{and} \quad Q = T^T \begin{pmatrix} Q_1 & O \\ O & Q_2 \end{pmatrix} T$$

which satisfy

$$PA + A^T P = Q.$$

Q is positive definite, and $V(x) = x^T P x > 0$ for points arbitrarily close to $x = 0$.

Let us study now the general case when A may have eigenvalues on the imaginary axis meanwhile A has eigenvalues with positive real parts. By a simple trick of shifting the imaginary axis, we suppose A has m eigenvalues with $\operatorname{Re}\lambda_j > \delta > 0$. Then, the matrix $A - \frac{\delta}{2}I$ has m eigenvalues in the open right-half plane, but no eigenvalues on the imaginary axis. Then, there exist $P = P^T$ and $Q = Q^T$ such that

$$P(A - \frac{\delta}{2}I) + (A - \frac{\delta}{2}I)^T P = Q,$$

where $V(x) = x^T P x$ is positive definite for points arbitrarily close to the origin. The derivative of $V(x)$ along the trajectories of (12.12) is given by

$$\dot{V}(x) = x^T(PA + A^T p)x + 2x^T P g(x)$$

$$= x^T[P(A - \frac{\delta}{2}I) + (A - \frac{\delta}{2}I)^T P]x + \delta x^T P x + 2x^T P g(x)$$

$$= x^T Q x + \delta V(x) + 2x^T P g(x).$$

In the set

$$U = \{x \in R^n \mid \|x\| \le r \text{ and } V(x) > 0\},$$

where r is chosen such that $\|g(x)\| \le \gamma \|x\|$ for $\|x\| < r$, $\dot{V}(x)$ satisfies

$$\dot{V}(x) \ge \lambda_{\min}(Q)\|x\|^2 - 2\|P\|\|x\|\|g(x)\| \ge (\lambda_{\min}(Q) - 2\gamma\|P\|)\|x\|^2,$$

which is positive definite if $\gamma < \dfrac{\lambda_{\min}(Q)}{2\|P\|}$. The origin of (12.12) is unstable. □

Remark 12.11 Theorem 12.5 does not say anything about the case when $\operatorname{Re}\lambda_j \le 0$ for all j, with $\operatorname{Re}\lambda_j = 0$ for some j. In this case, linearization fails to determine stability of equilibrium. Center manifold theory may apply.

Example 12.5 For $\dot{x} = ax^3$, where a is a parameter. Linearization yields

$$A = Df(0) = 3ax^2|_{x=0} = 0.$$

Hence, linearization fails. This failure is essential in the sense that $x = 0$ could be AS, stable, or unstable, depending on the value of the parameter a.

Take $V(x) = x^4 > 0$ as a Lyapunov function. $\dot{V}(x) = 4ax^6$. Then,

If $a < 0$, $\dot{V}(x) < 0 \Rightarrow x = 0$ is AS . If $a = 0$, $\dot{V}(x) \equiv 0 \Rightarrow x = 0$ is stable. If $a > 0$, $\dot{V}(x) > 0 \Rightarrow x = 0$ is unstable.

4. Summary

- Theorem 12.1-12.3 consist of the classical theory of Lyapunov method. LaSalle-Krosovskii Theorem is the starting of the modern one.

- GAS is more interesting for engineering application because it is no need for the estimation of a region of attraction, which is usually a tough work. However, GAS requirement is more demanding. In control, people hope to get (robustly) globally asymptotical stabilization by feedback (refer to feedback control), or moreover, to meet some additional optimized condition (refer to optimized control).

Homework

1. Study the stability of the pendulum equation with friction

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{k}{m}x_2 \end{cases}$$

by linearization.